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# Sufficient Conditions for Boundedness of Solutions of Linear Differential Equations

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In this paper we obtain sufficient conditions for boundedness of every solution and its derivatives of a linear differential equation. We will also show that our conditions are sharp.

Bounds for solutions of linear differential systems were obtained by Schwarz in [2] and [3]. Here we obtain sufficient conditions for boundedness of any solution and its derivatives up to a given order, for the linear differential equation,

$$y^{(n)}(z) + \sum_{k=1}^{n} p_k(z)\, y^{(n-k)}(z) = 0, \tag{1}$$

where the coefficients $p_1(z),\ldots, p_n(z)$ are regular functions in $|z| < 1$.

We get these results in the following two theorems:

THEOREM 1. *Let the coefficients $p_1(z),\ldots, p_n(z)$ of Eq.* (1) *be regular in $|z| < 1$. If*

$$\int_0^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{k-1}} | p_k(r_k e^{i\phi})|\, dr_k\,, \qquad k = 1, 2,\ldots, n, \tag{2}$$

*converge uniformly for $0 \leqslant \phi < 2\pi$, then every solution of Eq.* (1) *is bounded in $|z| < 1$ (here, by definition, $r_0 = 1$).*

THEOREM 2. *Let $p_1(z),\ldots, p_n(z)$ be regular functions in $|z| < 1$, and let $m$ be an integer, $1 \leqslant m < n$. If*

$$\int_0^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{k-1}} | p_k(r_k e^{i\phi})|\, dr_k\,, \qquad k = 1, 2,\ldots, n-m, \tag{3}$$

*and*

$$\int_0^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{n-m-1}} | p_s(r_{n-m} e^{i\phi})|\, dr_{n-m}\,, \qquad s = n-m+1,\ldots, n, \tag{4}$$



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*converge uniformly for* $0 \leqslant \phi < 2\pi$, *then every solution of Eq.* (1) *and its derivatives up to the order m, are bounded in* $|z| < 1$.

We note here that the above two theorems can be formulated as one, but it is easier to prove them separately.

The following two lemmas are used in the proof:

LEMMA 1. *Let* $f(z)$ *be a regular function in* $|z| < 1$. *If*

$$\int_0^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{s-1}} |f(r_s e^{i\phi})| \, dr_s , \tag{5}$$

*converges uniformly for* $0 \leqslant \phi < 2\pi$, *then the integral*

$$\int_0^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{s-1}} \left| f\left(te^{i\alpha} + \frac{t-t^2}{2} e^{i\beta}\right) \right| dt, \tag{6}$$

*converges uniformly for* $0 \leqslant \alpha, \beta < 2\pi$.

*Proof.* We transform (5) by substitution:

$$r_s e^{i\phi} = te^{i\alpha} + \frac{t-t^2}{2} e^{i\beta}, \tag{7}$$

where $t$, $\alpha$, $\beta$ are real numbers. Thus

$$r_s = t\left[1 + (1-t)\cos(\alpha-\beta) + \left(\frac{1-t}{2}\right)^2\right]^{1/2}, \tag{8}$$

and

$$dr_s = \frac{2 + \frac{1}{2}(1-2t)(1-t) + (2-3t)\cos(\alpha-\beta)}{2[1 + (1-t)\cos(\alpha-\beta) + ((1-t)/2)^2]^{1/2}} \, dt. \tag{9}$$

According to (9), for $0 \leqslant t \leqslant 1$

$$dr_s \geqslant \tfrac{1}{6} \, dt \tag{10}$$

(first we check (10) for $0 \leqslant t \leqslant \frac{2}{3}$, then for $\frac{2}{3} \leqslant t \leqslant 1$, and we also remark that

$$(a^2 + 2ab\cos\gamma + b^2)^{1/2} \leqslant a + b, \qquad \text{for } (a, b \geqslant 0).$$

By (8)

$$t = t(r_s, \alpha, \beta), \qquad t(0, \alpha, \beta) = 0,$$

and for $0 \leqslant t \leqslant 1$,

$$r_s \leqslant t\left(1 + \frac{1-t}{2}\right) \leqslant \frac{3t}{2}.$$

Therefore

$$t(r_s, \alpha, \beta) \geqslant \tfrac{3}{2} - (\tfrac{9}{4} - 2r_s)^{1/2}. \tag{11}$$

Take $a$, $0 \leqslant a \leqslant 1$. Then because of (7) and (10), and using the inequality

$$r_{s-1} \geqslant \tfrac{3}{2} - (\tfrac{9}{4} - 2r_{s-1})^{1/2}, \qquad 0 \leqslant r_{s-1} \leqslant 1,$$

we obtain

$$\begin{aligned}\int_a^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{s-2}} dr_{s-1} \int_0^{3/2-(9/4-2r_{s-1})^{1/2}} \left| f\left(te^{i\alpha} + \frac{t-t^2}{2}\, e^{i\beta}\right) \right| dt \\ \leqslant 6 \int_a^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{s-2}} dr_{s-1} \int_0^{r_{s-1}} |f(r_s e^{i\phi})|\, dr_s\, .\end{aligned} \tag{12}$$

Now, we define

$$v_{s-1} = \tfrac{3}{2} - (\tfrac{9}{4} - 2r_{s-1})^{1/2}, \qquad 0 \leqslant r_{s-1} \leqslant 1.$$

Then

$$dr_{s-1} = (\tfrac{3}{2} - v_{s-1})\, dv_{s-1}\, , \qquad 0 \leqslant v_{s-1} \leqslant 1.$$

Thus

$$\begin{aligned}&\int_0^{r_{s-2}} dr_{s-1} \int_0^{3/2-(9/4-2r_{s-1})^{1/2}} \left| f\left(te^{i\alpha} + \frac{t-t^2}{2}\, e^{i\beta}\right) \right| dt \\ &\geqslant \int_0^{3/2-(9/4-2r_{s-2})^{1/2}} \left(\frac{3}{2} - v_{s-1}\right) dv_{s-1} \int_0^{v_{s-1}} \left| f\left(te^{i\alpha} + \frac{t-t^2}{2}\, e^{i\beta}\right) \right| dt \\ &\geqslant \frac{1}{2} \int_0^{3/2-(9/4-2r_{s-2})^{1/2}} dv_{s-1} \int_0^{v_{s-1}} \left| f\left(te^{i\alpha} + \frac{t-t^2}{2}\, e^{i\beta}\right) \right| dt.\end{aligned}$$

Similarly, step by step, we obtain

$$\begin{aligned}\int_a^1 dv_1 \int_0^{v_1} dv_2 \cdots \int_0^{v_{s-2}} dv_{s-1} \int_0^{v_{s-1}} \left| f\left(te^{i\alpha} + \frac{t-t^2}{2}\, e^{i\beta}\right) \right| dt \\ \leqslant 6 \cdot 2^{s-1} \int_a^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{s-1}} |f(r_s e^{i\phi})|\, dr_s\, .\end{aligned} \tag{13}$$

Note, that in the proof of (13), in the final step we use the inequality

$$a \geqslant \tfrac{3}{2} - (\tfrac{9}{4} - 2a)^{1/2},$$

and hence

$$\int_a^1 dv_1 \cdots \int_0^{v_{s-1}} |f|\, dt \leqslant \int_{3/2-(9/4-2a)^{1/2}}^1 dv_1 \cdots \int_0^{v_{s-1}} |f|\, dt.$$

By assumption (5) of Lemma 1, for every $\epsilon$, $\epsilon > 0$, there is such a number $A(\epsilon) > 0$, that if $1 > a > A(\epsilon)$, then

$$6 \cdot 2^{s-1} \int_a^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{s-1}} |f(r_s e^{i\phi})| \, dr_s < \epsilon, \qquad \text{for } 0 \leqslant \phi < 2\pi. \tag{14}$$

The inequalities (13) and (14) complete the proof of Lemma 1.

LEMMA 2. *Let $f(z)$ be a regular function in $|z| < 1$. If*

$$J_{0,p} \overset{\text{def}}{=} \int_0^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{p-1}} |f(r_p e^{i\phi})| \, dr_p \tag{15}$$

*converges uniformly for $0 \leqslant \phi < 2\pi$, then*

$$J_{m,p} \overset{\text{def}}{=} \int_0^1 dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{p+m-1}} |f^{(m)}(r_{p+m} e^{i\phi})| \, dr_{p+m}, \qquad m = 1, 2, \ldots \tag{16}$$

*converge uniformly for $0 \leqslant \phi < 2\pi$.*

*Proof.* For $1 > r \geqslant r_1 \geqslant \cdots \geqslant r_p \geqslant 0$, we define

$$J(r) = \int_0^r dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{p-1}} dr_p \int_0^{r_p} |f'(r_{p+1} e^{i\phi})| \, dr_{p+1}. \tag{17}$$

It is well known (see Courant [1]) that $J(r)$ can be written as follows:

$$J(r) = \frac{1}{p!} \int_0^r (r - t)^p \, |f'(te^{i\phi})| \, dt. \tag{18}$$

Let $z = te^{i\phi}$, $0 < t < 1$. The function $f(z)$ is regular in $|z| < 1$, and therefore

$$f'(z) = \frac{1}{2\pi i} \int_{C_t} \frac{f(\zeta) \, d\zeta}{(\zeta - z)^2}, \tag{19}$$

where

$$C_t = \left\{ \zeta \mid |\zeta - z| = \frac{t - t^2}{2} \right\}.$$

Consequently, according to (19), if $r > t_0 > 0$ and $r > t \geqslant t_0$, then

$$|f'(te^{i\phi})| \leqslant \frac{1}{(1 - t) \, t_0 \pi} \int_0^{2\pi} \left| f\left(te^{i\phi} + \frac{t \mp t^2}{2} e^{i\psi}\right) \right| d\psi. \tag{20}$$

Furthermore, (18) gives us

$$J(r) = \frac{1}{p!} \left( \int_0^{t_0} (r - t)^p \, |f'(te^{i\phi})| \, dt + \int_{t_0}^r (r - t)^p \, |f'(te^{i\phi})| \, dt \right).$$

Thus, by (20)

$$\begin{aligned} J(r) &\leqslant \frac{1}{p!}\Big(\int_0^{t_0} (r-t)^p \, |f'(te^{i\phi})| \, dt \\ &\quad + \frac{1}{t_0\pi}\int_{t_0}^{r} \frac{(r-t)^p}{1-t} \, dt \int_0^{2\pi} \Big| f\Big(te^{i\phi} + \frac{t-t^2}{2} e^{i\psi}\Big)\Big| \, d\psi\Big) \\ &\leqslant \frac{1}{p!}\Big(\int_0^{t_0} (r-t)^p \, |f'(te^{i\phi})| \, dt \\ &\quad + \frac{1}{t_0\pi}\int_0^{2\pi} d\psi \int_{t_0}^{r} (r-t)^{p-1} \Big| f\Big(te^{i\phi} + \frac{t-t^2}{2} e^{i\psi}\Big)\Big| \, dt\Big). \end{aligned} \tag{21}$$

From assumption (15), using Lemma 1 and inequality (21), we conclude that $J(r)$ converges for $r \to 1$, uniformly for $0 \leqslant \phi < 2\pi$. So, we have proved (16) for $m = 1$, and by induction (16) holds for every positive integer $m$.

This completes the proof of Lemma 2. Now we are ready to prove Theorem 1.

*Proof of Theorem* 1. Let $y(z)$ be a solution of Eq. (1). From (1)

$$\begin{aligned} &\int_w^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} y^{(n)}(z_n) \, dz_n \\ &\quad = -\sum_{k=1}^{n} \int_w^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} p_k(z_n) \, y^{(n-k)}(z_n) \, dz_n . \end{aligned} \tag{22}$$

Hence, using integration by parts, we obtain

$$\begin{aligned} &\int_w^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} p_k(z_n) \, y^{(n-k)}(z_n) \, dz_n \\ &\quad = \int_w^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-2}} [p_k(z_{n-1}) \, y^{(n-k-1)}(z_{n-1}) - p_k(0) \, y^{(n-k-1)}(0)] \\ &\qquad - \int_w^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} p'_k(z_n) \, y^{(n-k-1)}(z_n) \, dz_n \\ &\quad = \sum_{s=0}^{n-k} (-1)^s \int_w^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{s+k-1}} p_k^{(s)}(z_{s+k}) \, y(z_{s+k}) \, dz_{s+k} + Q_k , \end{aligned} \tag{23}$$

where $Q_k$ is a polynomial. We denote

$$M(r, y) = \max_{|z|=r} |y(z)| = M(r).$$

and

$$\begin{aligned} &z = re^{i\phi}, \qquad z_j = r_j e^{i\phi}, \qquad j = 1, 2, \ldots, n, \\ &w = ae^{i\phi}, \qquad 0 < a \leqslant r < 1. \end{aligned} \tag{24}$$

Then from (23) and (24) we obtain

$$
\begin{aligned}
J_k &\overset{\text{def}}{=} \left| \int_a^r dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{n-1}} p_k(r_n e^{i\phi})\, y^{(n-k)}(r_n e^{i\phi})\, dr_n \right| \\
&\leqslant M(r) \sum_{s=0}^{n-k} \int_a^r dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{s+k-1}} | p_k^{(s)}(r_{s+k} e^{i\phi})|\, dr_{s+k} + A_k ,
\end{aligned} \tag{25}
$$

where $A_k$ is a positive constant.

According to assumption (2) of Theorem 1 and using Lemma 2, all the integrals in (25) converge uniformly for $0 \leqslant \phi < 2\pi$. Furthermore, for every $\epsilon_k$, $\epsilon_k > 0$, there exists such a number $a$, that in (25)

$$J_k < \epsilon_k M(r) + A_k , \qquad k = 1, 2,..., n. \tag{26}$$

We choose $a$ and $\epsilon_1, \epsilon_2 ,..., \epsilon_n$, such that

$$\sum_{k=1}^{n} \epsilon_k = q < 1. \tag{27}$$

Thus (22), (25), and (26) give us

$$| y(z)| \leqslant \sum_{k=1}^{n} J_k < M(r) \sum_{k=1}^{n} \epsilon_k + A, \tag{28}$$

where $| z | \leqslant r < 1$, and $A$ is a constant. So, by (27)

$$| y(z)| < qM(r) + A. \tag{29}$$

By (29)

$$M(r) < qM(r) + A.$$

or

$$M(r) < A/(1 - q) \qquad \text{for } 0 \leqslant r < 1. \tag{30}$$

This proves Theorem 1.

*Proof of Theorem* 2. First we will prove Theorem 2 for $m = 1$ (see (3) and (4)). From Eq. (1) we get

$$
\begin{aligned}
y'(z) &= \int_0^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-2}} y^{(n)}(z_{n-1})\, dz_{n-1} + P(z) \\
&= - \sum_{k=1}^{n} \int_0^z dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-2}} p_k(z_{n-1})\, y^{(n-k)}(z_{n-1})\, dz_{n-1} + P(z),
\end{aligned} \tag{31}
$$

where $P(z)$ is a polynomial. Denote

$$M(r, y') = \max_{|z|=r} | y'(z)| = M_1(r). \tag{32}$$

As in (26), we obtain

$$\left| \int_a^r dr_1 \cdots \int_0^{r_{n-2}} p_k(r_{n-1}e^{i\phi})\, y^{(n-k)}(r_{n-1}e^{i\phi})\, dr_{n-1} \right| < \epsilon_k M_1(r) + B_k . \tag{33}$$

Furthermore, Theorem 1 shows that every solution of Eq. (1) is bounded. So

$$J_n(r) \overset{\text{def}}{=} \int_a^r dr_1 \int_0^{r_1} dr_2 \cdots \int_0^{r_{n-2}} | p_n(r_{n-1}e^{i\phi})\, y(r_{n-1}e^{i\phi})|\, dr_{n-1} \tag{34}$$

is bounded for every $r$, $0 \leqslant r < 1$, and $0 \leqslant \phi < 2\pi$. By similar steps to those in (27), (28), (29) and using (33) and (34), we obtain (as in (30))

$$M_1(r) < \infty, \qquad r \in [0, 1).$$

Now, by induction, we obtain Theorem 2 for every $m$, $1 \leqslant m < n$.

Before we show that Theorem 1 is sharp, we wish to point out a particular case of this theorem:

THEOREM 1′. *Every solution of Eq.* (1) *is bounded in* $| z | < 1$, *if*

$$| p_k(z)| \leqslant \frac{L}{(1 - | z |)^{k-\alpha}}, \qquad | z | < 1, \quad k = 1, 2,..., n, \quad 0 \leqslant L < \infty, \quad \alpha > 0 \tag{35}$$

($p_1(z),..., p_n(z)$ *are regular functions in* $| z | < 1$).

It is clear that the inequalities (2) are satisfied, if (35) is fulfilled. The following example shows the sharpness of Theorem 1′ (and thus of Theorem 1). Let

$$y^{(n)}(z) + \frac{K}{(1 - z)^n}\, y(z) = 0, \tag{36}$$

where $K$ is a constant. It is known that $y = (1 - z)^\rho$ is a solution of Eq. (36), where $\rho$ is a root of the equation

$$P_n(\rho) \overset{\text{def}}{=} (-1)^n \rho(\rho - 1) \cdot \cdots \cdot (\rho - n + 1) + K = 0. \tag{37}$$

If $\rho_1, \rho_2, ..., \rho_n$ are all the roots (by their multiplicity) of (37), then we can write

$$P_n(\rho) = (-1)^n (\rho - \rho_1)(\rho - \rho_2) \cdot \cdots \cdot (\rho - \rho_n) = 0. \tag{38}$$

Hence

$$\rho_1 \rho_2 \cdot \cdots \cdot \rho_n = K. \tag{39}$$

Note that the coefficients of $P_n(\rho)$ in (38) are real. Thus, if we take $K$, $K < 0$ (no matter how small $| K |$ is), there exists at least one negative root for Eq. (37), say $\rho_1$.

Therefore the solution

$$y = (1 - z)^{\rho_1}$$

is not bounded in $|z| < 1$. The example (36) shows that the condition $\alpha > 0$ in (35) cannot be omitted (even if $L$ is arbitrarily small). We also remark, that when $\alpha = 0$, condition (2) in Theorem 1 does not hold.

Before we proceed to an example, which shows that Theorem 2 is also sharp, we formulate a particular case of Theorem 2:

THEOREM 2′. *Let $p_1(z),..., p_n(z)$ in Eq. (1) be regular functions in $|z| < 1$. Let $m$ be an integer, $1 \leqslant m < n$. If*

$$|p_k(z)| \leqslant \frac{L}{(1 - |z|)^{k-\alpha}}, \tag{40}$$

*for $k = 1, 2,..., n - m$, and*

$$|p_k(z)| \leqslant \frac{L}{(1 - |z|)^{n-m-\alpha}}, \tag{41}$$

*for $k = n - m + 1,..., n$ and $0 \leqslant L < \infty$, $\alpha > 0$, then all solutions of Eq. (1) and their derivatives up to the order $m$, are bounded in $|z| < 1$.*

We remark that there exist proofs of Theorems 1′ and 2′ which are independent of those of Theorems 1 and 2 (see [4]). These proofs enable us to extend Theorems 1′ and 2′ for Eq. (1) with only continuous coefficients in $(-1, 1)$.

The sharpness of Theorem 2′ (and thus of Theorem 2) for $n = 2$ is shown by the following example:

$$y''(z) + \frac{2}{1 - z^2} y(z) = 0. \tag{42}$$

Equation (42) satisfies condition (35) of Theorem 1′. Therefore all the solutions of Eq. (42) are bounded in $|z| < 1$. One of the solutions is

$$y = (1 - z^2) \log \frac{1 + z}{1 - z} + 2z$$

(another independent solution is $y = 1 - z^2$). But

$$y'(z) = -2z \log \frac{1 + z}{1 - z} + 4$$

is not bounded in $|z| < 1$. This shows that the condition $\alpha > 0$ in Theorem 2′ cannot be omitted.

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